Probabilistic Inequalities and Upper Probabilities in Quantum Mechanical Entanglement

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Abstract

In this paper we analyze the existence of joint probabilities for the Bell-type and GHZ entangled states. We then propose the usage of nonmonotonic upper probabilities as a tool to derive consistent joint upper probabilities for the contextual hidden variables. Finally, we show that for the extreme example of no error, the GHZ state allows for the definition of a joint upper probability that is consistent with the strong correlations.

1 Introduction

The issue of the completeness of quantum mechanics has been a subject of intense research for almost a century. One of the most influential papers is undoubtedly that of Einstein,

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Podolski and Rosen (1935), where, after analyzing entangled two-particle states, concluded that quantum mechanics could not be considered a complete theory. In 1963 John Bell showed that not only was quantum mechanics incomplete but, if one wanted a complete description of local reality, one would obtain correlations that are incompatible with the ones predicted by quantum mechanics. This happens because some quantum mechanical states do not allow for the existence of joint probability distributions of all the possible outcomes of experiments. If a joint distribution were to exist, then one could consistently create a local hidden variable that would factor this distribution. The nonexistence of local hidden variables that would "complete" quantum mechanics, and hence the nonexistence of joint probability distributions, was confirmed experimentally by Aspect, Dalibard, and Roger (1982). They showed, in a series of beautifully designed experiments, that an entangled photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle),$$

(where $|+-\rangle \equiv |+\rangle_A \otimes |-\rangle_B$ represents, for example, two photons A and B with helicity $+1$ and -1, respectively) violates the Clauser-Horne-Shimony-Holt form of Bell's inequalities, as predicted by quantum mechanical computations.

The nonexistence of joint probability distributions also comes into play in the consistent-history interpretation of quantum mechanics. In this interpretation, each sequence of properties for a given quantum mechanical system represents a possible history for this system, and a set of such histories is called a family of histories. A family of consistent histories is one that has a joint probability distribution for all possible histories in this family. One can easily show that quantum mechanics implies the nonexistence of such probability functions for some families of histories. Families of histories that do not have a joint probability distribution are called inconsistent histories.

Another important example, also related to the nonexistence of a joint probability distribution, is the famous Kochen-Specker theorem, which shows that a given hidden-variable theory that is consistent with the quantum mechanical results has to be contextual, i.e., the hidden variable has to depend on the values of the actual experimental settings, regardless of

how far apart the actual components of the experiment are located.

More recently, a marriage between Bell's inequalities and the Kochen-Specker theorem led to the Greenberger-Horne-Zeilinger (GHZ) theorem. The GHZ theorem shows that if one assumes that one can consistently assign values to the outcomes of a measurement before the measure is performed, a contradiction arises. Once again, having a complete data table would allow us to compute the joint probability distribution, and therefore no joint distribution exists that is consistent with quantum mechanical results.

Although it is sometimes remarked that all the above contradictions hold only for non-contextual hidden variable theories, a general proof of this is not available. In this paper, we propose the usage of nonmonotonic upper probabilities as a tool to derive consistent joint upper probabilities for the contextual hidden variables.

2 Upper Probabilities and Bell-type entanglement

We saw in the previous section that, for some cases, quantum mechanics does not allow the existence of a joint probability distribution for all the observables. However, if we weaken the probability axioms, Suppes and Zanotti (1991) proved that a consistent set of upper probabilities for the events can be found. Upper probabilities are defined in the following way. Let Ω be a nonempty set, F a boolean algebra on Ω, and P^* a real valued function on F. Then the triple (Ω, F, P^*) is an upper probability if for all ξ_1 and ξ_2 in F we have that

(i) $0 \leq P^*(\xi_1) \leq 1$,

(ii) $P^*(\emptyset) = 0$,

(iii) $P^*(\Omega) = 1$,

and if ξ_1 and ξ_2 are disjoint, i.e. $\xi_1 \cap \xi_2 = \emptyset$, then

(iv) $P^*(\xi_1 \cup \xi_2) \leq P^*(\xi_1) + P^*(\xi_2)$.

As we see, property (iv) weakens the standard additivity axiom for probability. Since monotonicity is one of the consequences of the standard probability axioms, it may be true for an

upper probability that

$$\xi_1 \subseteq \xi_2 \text{ and } P^*(\xi_1) > P^*(\xi_2).$$

Let us see how upper probabilities can be used to obtain joint upper-probability distributions. We start with Bell's observables, represented by the random variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. Each random variable corresponds to a different angle of measurement for the Stern-Gerlach apparatus (we follow the example in Suppes and Zanotti (1981)). Bell's thought experiment consisted of a two-particle system with an entangled spin state. Since each random variable corresponds to different spin orientations, we can only measure two of them at the same time. Additionally, Bell's states are such that the expected values of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are zero, and we have the constraint

$$P(\mathbf{X}=1) = P(\mathbf{Y}=1) = P(\mathbf{Z}=1) = \frac{1}{2}. \tag{1}$$

The question that Bell posed is whether we can fill the missing values of the data table in a way that is consistent with the marginal distributions given by quantum mechanics for the pairs of variables, that is, $E(\mathbf{XY})$, $E(\mathbf{XZ})$, $E(\mathbf{YZ})$. It is well known that for some sets of angles the joint probability exists, while for other sets of angles it does not. We can prove that the joint probability doesn't exist in the following way. We start with the values for the expectations given by Bell:

$$E(\mathbf{XY}) = -\frac{\sqrt{3}}{2}, \tag{2}$$

$$E(\mathbf{XZ}) = -\frac{\sqrt{3}}{2}, \tag{3}$$

$$E(\mathbf{YZ}) = -\frac{1}{2}. \tag{4}$$

The above expectations correspond to the angles between detectors set as $\widehat{XY} = 30^{\text{o}}$, $\widehat{YZ} = 30^{\text{o}}$, and $\widehat{XZ} = 60^{\text{o}}$. It follows from Suppes and Zanotti (1981) that a joint probability

distribution for $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ as above defined exist if and only if

$$-1 \leq E(\mathbf{XY}) + E(\mathbf{YZ}) + E(\mathbf{XZ}) \leq 1 + 2\min\{E(\mathbf{XY}), E(\mathbf{YZ}), E(\mathbf{XZ})\}. \tag{5}$$

Clearly, inequalities (5) are violated for expectations given by (2)–(4), and no joint probability distribution exists.

What changes with upper probabilities? The system of linear equations necessary for the existence of a joint distribution becomes a system of inequalities. This change makes it possible to obtain solutions to the system, and then upper probabilities that are consistent with the observed expectations (Suppes and Zanotti, 1991).

3 Bell-type inequalities for the GHZ state

As we saw in Section 2, the two-particle entangled state used by Einstein, Podolski, and Rosen has observables whose correlations cannot be explained by a joint probability distribution. In 1989, Greenberger, Horne and Zeilinger (GHZ) concocted a four-system entangled state that had two new features: the correlations were connected to path interference and the values of the observables seemed to lead to mathematical contradictions. The seemingly mathematical contradiction arose from an assumption of existence of a local hidden-variables theory that could explain the experimental outcomes predicted by quantum mechanics. Thus, GHZ proved that quantum mechanics is incompatible with hidden variables without using inequalities, but instead using perfect correlations. Their result is known as the GHZ theorem.

GHZ's argument, as stated by Mermin (1990a), goes as follows. We start with a three-particle entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|+\rangle_2|-\rangle_3 + |-\rangle_1|-\rangle_2|+\rangle_3), \tag{6}$$

where $\hat{\sigma}_{iz}|\pm\rangle_i = \pm|\pm\rangle$, and $\hat{\sigma}_{iz}$ is the spin operator in the $\hat{\mathbf{z}}$ direction on the Hilbert space of

the i-th particle. This state is an eigenstate of the following spin operators:

$$\hat{\mathbf{A}} = \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}, \quad \hat{\mathbf{B}} = \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}, \tag{7}$$

$$\hat{\mathbf{C}} = \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}, \quad \hat{\mathbf{D}} = \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}. \tag{8}$$

If we compute the expected values for the above correlations, we obtain at once that $E(\hat{\mathbf{A}}) = E(\hat{\mathbf{B}}) = E(\hat{\mathbf{C}}) = 1$ and $E(\hat{\mathbf{D}}) = -1$. That these correlations present a problem can be seen from the following theorem (a simplified version of the theorem found in Suppes, de Barros, and Oas (1998)).

Theorem 1 Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three ± 1 random variables and let

(i) $E(\mathbf{A}) = E(\mathbf{B}) = E(\mathbf{C}) = 1$,

(ii) $E(\mathbf{ABC}) = -1$.

Then **(i)** and **(ii)** imply a contradiction.

Proof: By definition

$$E(a) = P(a) - P(\overline{a}), \tag{9}$$

where we use the notation $P(a) = P(\mathbf{A} = 1)$ and $P(\overline{a}) = P(\mathbf{A} = -1)$. Since $0 \leq P(a)$, $P(\overline{a}) \leq 1$, it follows at once from (i) that

$$P(a) = 1, \tag{10}$$

and

$$P(\overline{a}) = 0. \tag{11}$$

Similarly, from (ii) and (iii),

$$P(b) = P(c) = 1. \tag{12}$$

$$P(\overline{b}) = P(\overline{c}) = 0. \tag{13}$$

Using again the definition of expectation and the inequalities $P(\overline{a}bc) \leq P(\overline{a}) = 0$, etc., we

have

$$
\begin{aligned}
E(\mathbf{ABC}) &= P(abc) + P(\overline{a}\overline{b}c) + P(a\overline{b}\overline{c}) + P(\overline{a}b\overline{c}) \\
&\quad -[P(\overline{a}bc) + P(a\overline{b}c) + P(ab\overline{c}) + P(\overline{a}\overline{b}\overline{c})] \qquad (14)\\
&= 1,
\end{aligned}
$$

from (10) and (13), since all but the first term on the right are 0. Thus, by conservation of probability, $P(abc) = 1$ and the last line follows. But (14) contradicts (ii). $\diamondsuit$

Of course, the above theorem assumes the existence of an underlying joint probability distribution. The relationship between the above theorem and the existence of hidden variables can be illustrated by the following. Let us now suppose that the value of the spin for each particle is dictated by a hidden variable λ, and let us call this value $s_{ij}(\lambda)$, where $i = 1...3$ and $j = x, y$. Because spin measurements on each particle can be separated by a space-like interval, we have that

$$
\begin{aligned}
E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) &= (s_{1x}s_{2y}s_{3y})(s_{1y}s_{2x}s_{3y})(s_{1y}s_{2y}s_{3x}) \qquad (15)\\
&= s_{1x}s_{2x}s_{3x}(s_{1y}^2 s_{2y}^2 s_{3y}^2). \qquad (16)
\end{aligned}
$$

Since the $s_{ij}(\lambda)$ can only be 1 or -1, we obtain

$$
E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = s_{1x}s_{2x}s_{3x} = E(\hat{\mathbf{D}}). \qquad (17)
$$

But (15) implies that $E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = 1$, whereas (17) implies $E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = E(\hat{\mathbf{D}}) = -1$. It should be evident from the above derivation that we could avoid contradictions if we allow the value of λ to depend on the experimental setup, i.e., if we allow λ to be a contextual hidden variable. In other words, what the GHZ theorem proves is that non-contextual hidden variables cannot reproduce quantum mechanical predictions.

One of the striking characteristics of GHZ's example is that a contradiction between quantum mechanics and a hidden-variable theory comes from probability one (or zero) events. This, however, leads to a problem. How can we experimentally verify predictions based on correlation-one events given that experimentally we cannot obtain perfectly correlated events?

This problem was also present in Bell's original paper, where he didn't consider experimental errors. To "avoid Bell's experimentally unrealistic restrictions", Clauser, Horne, Shimony and Holt (1969) derived a new set of inequalities that would take into account imperfections in the measurement process. However, Bell's inequalities are quite different from the GHZ case, where it is *necessary* to have experimentally unrealistic perfect correlations.

It is important to note that if we could measure all the random variables simultaneously, we would have a joint distribution. The existence of a joint probability distribution is a necessary and sufficient condition for the existence of a hidden variable (Suppes and Zanotti (1991)). Hence, if the quantum mechanical GHZ correlations are obtained, then no hidden variable exists. However, this abstract version of the GHZ theorem still involves probability-one statements. On the other hand, the correlations present in the GHZ state are so strong that even if we allow for experimental errors, the non-existence of a joint distribution, or, equivalently (as shown by Suppes and Zanotti (1991)) the non-existence of a hidden variable, can still be verified, as we now proceed to show. We follow de Barros and Suppes (2001). We start by defining the ±1-valued random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ corresponding to the outcomes of spin measurements. The random variable representing the outcomes of $\hat{\sigma}_{1x}$ is $\mathbf{X}_1$, $\hat{\sigma}_{2x}$ is $\mathbf{X}_2$, $\hat{\sigma}_{1y}$ is $\mathbf{Y}_1$, and so on. Before we derive the inequalities, we note that if we could measure all the random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ simultaneously, we would have a joint probability distribution. The existence of a joint probability distribution is a necessary and sufficient condition for the existence of a noncontextual hidden variable (Suppes and Zanotti, 1991). Hence, if the quantum mechanical GHZ correlations are obtained, then no such hidden variable exists. However, Theorem 1 still involves probability-one statements. On the other hand, the quantum mechanical correlations present in the GHZ state are so strong that even if we allow for experimental errors, the non existence of a joint distribution can still be verified, as we show in the following theorem, which, as we said above, extends the results in de Barros and Suppes (2000).

Theorem 2 Let $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, be six ±1 random variables. Then, there exists a joint probability distribution for all six random variables if and only if the following

inequalities are satisfied:

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq -E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) - E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) - E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2.$$

Proof: The argument is similar to the one found in de Barros and Suppes (2000). To simplify, we use a notation where $x_1y_2y_3$ means $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3 = 1$, $\overline{x_1y_2y_3}$ means $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3 = -1$. We prove first that the existence of a joint probability distribution implies the four inequalities. Then, we have by an elementary probability computation that

$$\begin{aligned} P(x_1y_2y_3) &= P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) \\ &\quad + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) \end{aligned}$$

and

$$\begin{aligned} P(\overline{x_1y_2y_3}) &= P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ &\quad + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}), \end{aligned}$$

with similar equations for $\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3$ and $\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3$. But

$$\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 = (\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3),$$

and so we have that

$$\begin{aligned} P(x_1x_2x_3) &= P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ &\quad + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) \end{aligned}$$

and

$$\begin{aligned}
P(\overline{x_1x_2x_3}) &= P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\
&\quad + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}).
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
F &= 2[P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\
&\quad + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3})] \\
&\quad - 2[P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\
&\quad + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3)}],
\end{aligned}$$

where F is defined by

$$F = E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3).$$

Since all probabilities are non-negative and sum to ≤ 1, we infer the first inequality at once. The derivation of the other inequalities is similar.

Now for the sufficiency part. First, we assume the symmetric case where

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 2p - 1, \tag{18}$$

and

$$E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -(2p - 1). \tag{19}$$

Then, the first inequality yields

$$\frac{1}{4} \leq p \leq \frac{3}{4}, \tag{20}$$

while the other ones yield

$$0 \leq p \leq 1. \tag{21}$$

Since $\mathbf{X}_i$ and $\mathbf{Y}_i$ are ± 1 random variables, p has to belong to the interval $[0,1]$, and inequality (21) doesn't add anything new. We will prove the existence of a joint probability distribution for this symmetric case by showing that, given any p, $\frac{1}{4} \leq p \leq \frac{3}{4}$, we can assign values to the atoms that have the proper marginal distributions.

The probability space for $\mathbf{X}_i$ and $\mathbf{Y}_i$ has 64 atoms. It is difficult to handle a problem of this size, so we will assume some further symmetries to reduce the problem. First, we introduce the following notation: if a group of symbols is between square brackets, all the possible permutations of the bar symbol is considered. For example, $a_5 = P([\bar{x}_1 x_2 x_3] y_1 y_2 y_3)$ means that $P(\bar{x}_1 x_2 x_3 y_1 y_2 y_3) = a_5$, $P(x_1 \bar{x}_2 x_3 y_1 y_2 y_3) = a_5$, and $P(x_1 x_2 \bar{x}_3 y_1 y_2 y_3) = a_5$. Then, the number of independent values for the probabilities of atoms in the problem is reduced to the following 16: $a_1 = P(x_1 x_2 x_3 y_1 y_2 y_3)$, $a_2 = P(x_1 x_2 x_3 \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_3 = P(x_1 x_2 x_3 [\bar{y}_1 y_2 y_3])$, $a_4 = P(x_1 x_2 x_3 [\bar{y}_1 \bar{y}_2 y_3])$, $a_5 = P([\bar{x}_1 x_2 x_3] y_1 y_2 y_3)$, $a_6 = P([\bar{x}_1 x_2 x_3] \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_7 = P([\bar{x}_1 x_2 x_3][\bar{y}_1 y_2 y_3])$, $a_8 = P([\bar{x}_1 x_2 x_3][\bar{y}_1 \bar{y}_2 y_3])$, $a_9 = P([\bar{x}_1 \bar{x}_2 x_3][\bar{y}_1 y_2 y_3])$, $a_{10} = P([\bar{x}_1 \bar{x}_2 x_3][\bar{y}_1 \bar{y}_2 y_3])$, $a_{11} = P([\bar{x}_1 \bar{x}_2 x_3] y_1 y_2 y_3)$, $a_{12} = P([\bar{x}_1 \bar{x}_2 x_3] \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_{13} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 [\bar{y}_1 y_2 y_3])$, $a_{14} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 [\bar{y}_1 \bar{y}_2 y_3])$, $a_{15} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 y_1 y_2 y_3)$, $a_{16} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{y}_1 \bar{y}_2 \bar{y}_3)$.

These new added symmetries reduce the problem from 64 to 16 variables. The atoms have to satisfy various sets of equations. The first set comes just from the requirement that $E(\mathbf{X}_i) = E(\mathbf{Y}_i) = 0$, for $i = 1, 2, 3$, but two of the six equations are redundant, and so we are left with the following four.

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 + a_5 + a_6 + 3a_7 + 3a_8 - 3a_9 & \\ -3a_{10} - a_{11} - a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= 0, \end{aligned} \tag{22}$$

$$\begin{aligned} a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9 & \\ -3a_{10} + 3a_{11} - 3a_{12} + a_{13} - a_{14} + a_{15} - a_{16} &= 0, \end{aligned} \tag{23}$$

$$a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9$$

$$-3a_{10} + 3a_{11} - 3a_{12} - a_{13} + a_{14} + a_{15} - a_{16} = 0, \tag{24}$$

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 - a_5 - a_6 - 3a_7 - 3a_8 + 3a_9 & \\ +3a_{10} + a_{11} + a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= 0, \end{aligned} \tag{25}$$

where (22) comes from $E(\mathbf{X}_1) = 0$, (23) from $E(\mathbf{X}_2) = 0$, (24) from $E(\mathbf{Y}_1) = 0$, and (25) from $E(\mathbf{Y}_2) = 0$. The triple expectations also imply

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 - 3a_5 - 3a_6 - 9a_7 - 9a_8 + 9a_9 & \\ +9a_{10} + 3a_{11} + 3a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= -2p + 1, \end{aligned} \tag{26}$$

$$\begin{aligned} a_1 + a_2 - a_4 - a_4 + a_5 + a_6 - a_7 - a_8 + a_9 & \\ +a_{10} - a_{11} - a_{12} + a_{13} + a_{14} - a_{15} - a_{16} &= 2p - 1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} a_1 - a_2 - 3a_3 + 3a_4 + 3a_5 - 3a_6 - 9a_7 + 9a_8 - 9a_9 & \\ +9a_{10} + 3a_{11} - 3a_{12} - 3a_{13} + 3a_{14} + a_{15} - a_{16} &= 2p - 1. \end{aligned} \tag{28}$$

Finally, the probabilities of all atoms have to sum to one, yielding the last equation

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 + 3a_5 + 3a_6 + 9a_7 + 9a_8 + 9a_9 & \\ +9a_{10} + 3a_{11} + 3a_{12} + 3a_{13} + 3a_{14} + a_{15} + a_{16} &= 1. \end{aligned} \tag{29}$$

Even with the symmetries reducing the problem to 16 variables, we still have an infinite number of solutions that satisfy equations (22)–(29). Since it is very hard to exhibit a general solution for (22)–(29) and the constraints $0 \le a_i \le 1$, $i = 1 \ldots 16$, we will just show that particular solutions exist for an arbitrary p satisfying the inequality (20). To do so, we will

divide the problem into two parts: one where we will exhibit an explicit solution for the atoms $a_1, \ldots, a_{16}$ that form a proper probability distribution for $p \in [\frac{1}{4}, \frac{1}{2}]$, and another explicit solution for $p \in [\frac{1}{2}, \frac{3}{4}]$.

It is easy to verify that, given an arbitrary p in $[\frac{1}{4}, \frac{1}{2}]$, the following set of values constitute a solution of equations (22)–(29): $a_1 = 0$, $a_2 = -\frac{1}{2} + 2p$, $a_3 = \frac{1}{4} - \frac{1}{2}p$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, $a_{12} = \frac{1}{4} - \frac{1}{2}p$, $a_{13} = 0$, $a_{14} = 0$, $a_{15} = p$, $a_{16} = 0$. For p in $[\frac{1}{2}, \frac{3}{4}]$ the following set of values constitute a solution of equations (22)–(29): $a_1 = -\frac{1}{8} + \frac{1}{2}p$, $a_2 = 0$, $a_3 = \frac{3}{8} - \frac{1}{2}p$, $a_4 = 0$, $a_5 = -\frac{5}{24} + \frac{1}{3}p$, $a_6 = -\frac{1}{24} + \frac{1}{6}p$, $a_7 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = \frac{1}{8}$, $a_{15} = \frac{3}{8} - \frac{1}{2}p$, $a_{16} = 0$. So, for p satisfying the inequality $\frac{1}{4} \le p \le \frac{3}{4}$ we can always construct a probability distribution for the atoms consistent with the marginals, and this concludes the proof. $\diamondsuit$

We note that the form of the inequalities of Theorem 2 is actually that of the Clauser et al. (1969) for the Bell case, when the Bell binary correlations are replaced by the GHZ triple correlations. The inequalities from Theorem 2 immediately yield the following.

Corollary Let $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \le i \le 3$, be six ± 1 random variables, and let

(i) $E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 1 - \varepsilon$,

(ii) $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -1 + \varepsilon$,

$\varepsilon \in [0, 1]$. Then there cannot exist a joint probability distribution of $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \le i \le 3$, satisfying (i) and (ii) if $\varepsilon < \frac{1}{2}$.

Proof. If a joint probability exists, then

$$-2 \le E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \le 2.$$

But

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = 4 - 4\varepsilon,$$

and the inequality is satisfied only if $\varepsilon \ge \frac{1}{2}$. Hence, if $\varepsilon < \frac{1}{2}$ no joint probability exists. $\diamondsuit$

In the Corollary, ε may represent, for instance, a deviation from the predicted quantum mechanical correlations due to experimental errors. So, we see that to prove the nonexistence of a joint probability distribution for the GHZ experiment, we do not need to have perfect measurements and 1 or -1 correlations. In fact, from the above inequalities, it should be clear that any experiment that satisfies the strong symmetry of the Corollary and obtains a correlation for the triples stronger than 0.5 (and -0.5 for one of them) cannot have a joint probability distribution.

It is worth mentioning at this point that the inequalities derived in Theorem 2 have a completely different origin than do Bell's inequalities. The inequalities of Theorem 2 are not satisfied by a particular model, but they just accommodate the theoretical conditions in GHZ to possible experimental deviations. Also, Theorem 2 does not rely on any "enhancement" hypothesis to reach its conclusion. Thus, with this reformulation of the GHZ theorem it is possible to use strong, yet imperfect, experimental correlations to prove that a hidden-variable theory is incompatible with the experimental results.

4 Upper probabilities and the GHZ state

We now analyze the existence of upper probabilities for the GHZ state. The following theorem states our main result.

Theorem 3 Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three ± 1 random variables and let

(i) $E^*(\mathbf{A}) = E(\mathbf{A}) = 1$,

(ii) $E^*(\mathbf{B}) = E(\mathbf{B}) = 1$,

(iii) $E^*(\mathbf{C}) = E(\mathbf{C}) = 1$,

(iv) $E^*(\mathbf{ABC}) = E(\mathbf{ABC}) = -1$.

Then, there exists an upper joint probability distribution that is compatible with expectations (i)–(iv).

Proof: We prove the theorem by explicitly providing an upper joint probability distribution. Let

$$p^*(abc) = p^*\left(\overline{a}\overline{b}\overline{c}\right) = 1 \tag{30}$$

and

$$p^*(\overline{a}bc) = p^*\left(a\overline{b}c\right) = p^*(ab\overline{c}) = p^*\left(a\overline{b}\overline{c}\right) = p^*(\overline{a}b\overline{c}) = p^*\left(\overline{a}\overline{b}c\right) = 0. \tag{31}$$

Since $E^*(\mathbf{A}) = E^*(\mathbf{B}) = E^*(\mathbf{C}) = 1$, it follows that

$$p^*(a) = p^*(b) = p^*(c) = 1 \tag{32}$$

and

$$p^*(\overline{a}) = p^*\left(\overline{b}\right) = p^*(\overline{c}) = 0. \tag{33}$$

Next, let us consider the events

$$a = abc \cup a\overline{b}c \cup ab\overline{c} \cup a\overline{b}\overline{c}, \tag{34}$$

$$\overline{a} = \overline{a}bc \cup \overline{a}\overline{b}c \cup \overline{a}b\overline{c} \cup \overline{a}\overline{b}\overline{c}, \tag{35}$$

and similarly for b, $\overline{b}$, c, and $\overline{c}$. From (34) and the subadditive properties of the upper distributions must hold, and we have

$$p^*(a) \leq p^*(abc) + p^*\left(a\overline{b}c\right) + p^*(ab\overline{c}) + p^*\left(a\overline{b}\overline{c}\right).$$

Using (30), (31), and (32), the above inequality becomes

$$1 \leq 1 + 0 + 0 + 0,$$

consistent with the joint. For (35), the subadditive properties requires

$$p^*(\overline{a}) \leq p^*(\overline{a}bc) + p^*\left(\overline{a}\overline{b}c\right) + p^*(\overline{a}b\overline{c}) + p^*\left(\overline{a}\overline{b}\overline{c}\right).$$

Using (30), (31), and (33), the above inequality becomes

$$0 \leq 0 + 0 + 0 + 1,$$

also consistent. Similar computations follow for b, $\overline{b}$, c, and $\overline{c}$.

Going back to the expectation, we are given

$$E\left(\mathbf{ABC}\right) = -1,$$

or

$$E^{*}\left(\mathbf{ABC}\right) = 1 \cdot p^{*}\left(abc \cup \overline{a}\overline{b}c \cup \overline{a}b\overline{c} \cup a\overline{b}\overline{c}\right) + (-1) \cdot p^{*}\left(\overline{a}bc \cup a\overline{b}c \cup ab\overline{c} \cup \overline{a}\overline{b}\overline{c}\right). \tag{36}$$

From (36)

$$p^{*}\left(abc \cup \overline{a}\overline{b}c \cup \overline{a}b\overline{c} \cup a\overline{b}\overline{c}\right) = 0, \tag{37}$$

and from (30), (31), (37), and the subadditive properties,

$$\begin{aligned} p^{*}\left(abc \cup \overline{a}\overline{b}c \cup \overline{a}b\overline{c} \cup a\overline{b}\overline{c}\right) &\leq p^{*}\left(abc\right) + p^{*}\left(\overline{a}\overline{b}c\right) + p^{*}\left(\overline{a}b\overline{c}\right) + p^{*}\left(a\overline{b}\overline{c}\right) \\ 0 &= 1 + 0 + 0 + 0. \end{aligned}$$

Also, from (36),

$$p^{*}\left(\overline{a}bc \cup a\overline{b}c \cup ab\overline{c} \cup \overline{a}\overline{b}\overline{c}\right) = 1, \tag{38}$$

and from (30), (31), (38), and the subadditive properties,

$$\begin{aligned} p^{*}\left(\overline{a}bc \cup a\overline{b}c \cup ab\overline{c} \cup \overline{a}\overline{b}\overline{c}\right) &\leq p^{*}\left(\overline{a}bc\right) + p^{*}\left(a\overline{b}c\right) + p^{*}\left(ab\overline{c}\right) + p^{*}\left(\overline{a}\overline{b}\overline{c}\right) \\ 1 &= 0 + 0 + 0 + 1. \end{aligned}$$

Thus, we complete the check of all probabilities necessary for consistency. The remaining events can easily be assigned upper probabilities that satisfy the axioms of upper probabilities.

♢

5 Conclusions

To apply the upper probabilities to the GHZ theorem, we gave a probabilistic random variable version of it. We then showed that, if we use upper probabilities, some of the lemmas used to derive GHZ do not hold anymore, and hence the inconsistencies cannot be proved to exist from the upper probabilities. But upper probabilities are a natural way to deal with contextual problems in statistics. For example, in pools in social sciences upper probabilities can be applied, as the results of the responses depend on the context of the pool. This contextuality in the GHZ derivation was shown in the previous section. Therefore, generic contextual hidden variable theories do not result in any logical contradictions, as often claimed, since the mathematical contradictions derived in GHZ do not appear when we weaken the requirements for probabilities, allowing them to be upper probabilities instead.

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